Math 245C Lecture 17 Notes

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1 Orthonormal Basis of L^2 and the Fourier Transform

1.1 An orthonormal basis of $L^2(\mathbb{T})$

If $\xi \in \mathbb{R}^n$, we define $E_\xi : \mathbb{R}^n \to \mathbb{C}$ by $E_\xi(x) = e^{2\pi i x \cdot \xi}$, where $x \cdot \xi = \sum_{i=1}^n x_i \xi_i$. Let $\mathcal{E} = \{E_k : k \in \mathbb{Z}^n\}$.

Proposition 1.1. *$\mathcal{E}$ separates points in $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$: For $a, b \in \mathbb{T}^n$, if $E_k(a) = E_k(b)$ for all $k \in \mathbb{Z}^n$, then $a = b$.*

Proof. Assume $E_k(a) = E_k(b)$. Then $e^{2\pi i a \cdot k} = e^{2\pi i b \cdot k}$, so $e^{2\pi i (b-a)\cdot k} = 1$. So $\cos(2\pi(b - a) \cdot k) = 1$, and $\sin(2\pi(b-a)\cdot k) = 0$. This means $(b-a)\cdot k \in \mathbb{Z}$, and this holds for all $k \in \mathbb{Z}^k$. In particular, taking $k = (0, \dots, 0, 1, 0, \dots, 0)$, we conclude that $e^{2\pi i (b_j - a_j)} = 0$ for $j = 1, \dots, n$. So $b_j - a_j \in \mathbb{Z}$, which means that $b_j - a_j = 0$ (since $a, b \in [0,1)^n$ and $|a_j - b_j| < 1$). $\square$

Theorem 1.1. *The collection $\mathcal{E}$ is an orthonormal basis of $L^2(\mathbb{T}^n)$ for the inner product $\langle f, g \rangle = \int_{\mathbb{T}^n} f(x)\overline{g(x)}\, dx = \int_{[0,1]^n} f(x)\overline{g(x)}\, dx$.*

Proof. Let $k, \ell, \in \mathbb{Z}^n$. Then

$$\langle E_k, E_\ell \rangle = \int_{[0,1]^n} e^{2\pi i (k-\ell)\cdot x}\, dx = \prod_{j=1}^n \int_0^1 e^{2\pi i (k_j - \ell_j) x_j}\, dx_j = \begin{cases} 1 & k = \ell \\ 0 & k \neq \ell. \end{cases}$$

So $\mathcal{E}$ is orthonormal.

It remains to show that $\mathcal{E}$ spans a dense subset of $L^2(\mathbb{T}^n)$. Let $A = \{\sum_{k \in \Lambda} \lambda_k E_k : \Lambda \subseteq \mathbb{Z}^n \text{ is finite}, \lambda_k \in \mathbb{C}\}$. Since $E_k E_\ell \in \mathcal{E}$ for any $k, \ell \in \mathbb{Z}^n$, one checks that $\mathcal{A}$ is an algebra in $C(\mathbb{T}^n)$. Since $\mathbb{C} = \{\lambda E_0 : \lambda \in \mathbb{C}\}$, we conclude that $\mathcal{A}$ contains the constant functions. By the Stone-Weierstrass theorem, $\mathcal{A}$ is dense in $C(\mathbb{T}^n)$ for the uniform norm. If $f \in L^2(\mathbb{T}^n)$ and $\varepsilon > 0$, there exists $g \in C(\mathbb{T}^n)$ such that $\|f - g\|_2 < \varepsilon/2$. Choose $h \in \mathcal{A}$ such that $\|g - h\|_u \,\mathrm{eq}\, \varepsilon/2$. Then $\|g - h\|_2 \leq \|g - h\|_u$, since $m(\mathbb{T}) = 1$. Consequently, $\|f - h\|_2 < \varepsilon$. This proves that $\mathcal{A}$ is dense in $L^2(\mathbb{T})$. $\square$

1.2 The Fourier transform

Remark 1.1. Let $f \in L^2(\mathbb{T}^n)$. Then

$$f = \sum_{k \in \mathbb{Z}^n} \langle f, E_k \rangle E_k, \qquad \|f\|_2^2 = \sum_{k \in \mathbb{Z}^n} |\langle f, E_k \rangle|^2.$$

Set

$$\widehat{f}_k = \langle f, E_k \rangle, \qquad \widehat{f} = (\widehat{f}_k)_{k \in \mathbb{Z}^n}.$$

We have a map $\Lambda : L^2(\mathbb{T}^n) \to \ell^2(\mathbb{Z}^n)$ sending $f \mapsto \widehat{f}$. This is an isometry because this relation gives $\|f\|_2 = \|\widehat{f}\|_2$ (Parseval's identity).

Remark 1.2. Observe that if $f \in L^1(\mathbb{T}^n)$, since $E_k \in L^\infty(\mathbb{T}^n)$, we have $fE_k \in L^1(\mathbb{T}^n)$, and so $\widehat{f}_k$ is still well defined. Note that

$$|\widehat{f}_k| = \left| \int_{\mathbb{T}^n} f(x) e^{2\pi i k \cdot x} \, dx \right| \leq \|f\|_1.$$

In other words,

$$\|\widehat{f}\|_{\ell^\infty(\mathbb{Z}^n)} \leq \|f\|_1.$$

Theorem 1.2. *Let $1 < p < 2$, and let $q = p/(p-1)$ be the conjugate exponent. Then the Fourier transform Λ extends to a linear map $\Lambda : L^p(\mathbb{R}^n) \to \ell^q(\mathbb{Z}^n)$ such that*

$$\|\widehat{f}\|_{\ell^q(\mathbb{Z}^n)} \leq \|f\|_{L^p(\mathbb{T}^n)}.$$

Proof. We want to apply the Riesz-Thorin theorem. Set $p_0 = 2$ and $p_1 = 1$, so $q_0 = 2$ and $q_1 = \infty$. Set $t = 2/p - 1 \in (0, 1)$, and set

$$\frac{1}{p_t} := \frac{t}{p_0} + \frac{1-t}{p_0} = \frac{1}{p}, \qquad \frac{1}{q_t} := \frac{t}{q_0} + \frac{1-t}{q_0} = \frac{1}{q}.$$

By the Riesz-Thorin interpolation theorem,

$$\|\widehat{f}\|_{\ell^q(\mathbb{Z}^n)} \leq M_1^t M_0^{1-t} \|f\|_{L^p(\mathbb{T}^n)} = \|f\|_{L^p(\mathbb{T}^n)},$$

as $M_0 = M_1 = 1$. □